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Confidence Intervals for the Ratio between Two Means of Birnbaum-Saunders Distributions

Ratasukharom, N.[1], Niwitpong, S. A.[1], and Niwitpong, S.*[1]

[1] *Department of Applied Statistics, Faculty of Applied Sciences, King Mongkut's University of Technology North Bangkok, Thailand*

E-mail: suparat.n@sci.kmutnb.ac.th
**Corresponding author*

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Abstract

In this study, we introduced several methods to estimate confidence intervals for the ratio between means of Birnbaum-Saunders (BirSau) distributions. These methods encompassed the generalized confidence interval (GCI), bootstrap confidence interval (BCI), percentile bootstrap confidence interval (PBCI), Bayesian credible interval (BayCrI), and the highest posterior density (HPD). We conducted a Monte Carlo simulation to assess their performance, focusing on coverage probabilities and average widths. The simulation results revealed that the HPD method consistently delivered strong results for small and medium sample sizes across various scenarios. However, in the case of larger samples, the BCI method emerged as the most effective option. It was observed that as the sample size increased, the average widths of the proposed confidence intervals tended to decrease. Furthermore, we applied these methods to establish confidence intervals for the ratio between the means of wind speed datasets originating from two separate industrial regions in Thailand: Samut Prakan and Rayong province. The results obtained from this real-data application closely aligned with the findings derived from our simulation results.

Keywords: generalized confidence interval; bootstrap; bayesian; wind speed.

1 Introduction

Since 2011, Thailand has grappled with a pressing air pollution issue, primarily around fine dust particles measuring less than 2.5 microns ($PM_{2.5}$). This predicament is particularly acute in major urban centers and industrial zones. It stems from a multitude of sources, encompassing emissions from transportation, the incineration of agricultural residue after the harvest season, the influx of smog from neighboring countries, and the expansion of industrial operations within the country. The repercussions of this pollution are substantial, with a disproportionate impact on the health of various vulnerable demographics, including children, pregnant women, the elderly, and individuals living with chronic ailments. A study conducted by Paneangtong et al. [24] revealed that people living in industrial areas had a higher rate of respiratory illnesses compared to those in less industrialized regions. The severity of the $PM_{2.5}$ issue intensifies, especially during the dry season, which occurs from December to April each year. This is primarily due to the dry and still air, causing fine dust particles to remain suspended in the atmosphere for extended periods. Amnuaylojaroen et al. [2] conducted a study revealing an inverse relationship between $PM_{2.5}$ and wind speed. To be more specific, an increase in wind speed is associated with a decrease in the concentration of $PM_{2.5}$ in the atmosphere. Mohammadi et al. [21] explored and evaluated the use of the two-parameter Birnbaum-Saunders (BirSau) distribution to analyze the wind speed distribution in a long-term times series of recorded wind speed data from ten different stations. Their findings highlighted the effective performance of the BirSau distribution across all ten stations. We are interested in estimating wind speed because it provides crucial information about the current $PM_{2.5}$ concentration based on past data. The daily wind speed data for the industrial regions of Samut Prakan and Rayong provinces adhere to the assumptions of the BirSau distribution. In this study, estimating the ratio between the means of wind speeds in these two areas serves as the initial point. The mean or expected value not only represents an average value adjusted according to the probability distribution but is also regarded as the parameter of interest, signifying a measure of the population's central tendency [5]. Additionally, the ratio between the means of the two datasets can be utilized to quantify their difference.

Birnbaum and Saunders introduced the BirSau distribution [4]. As a failure-time distribution, a BirSau distribution describes the total period until the damage produced by the development and growth of a dominant crack approaches the threshold at which the component fails. The BirSau distribution is characterized by two fundamental parameters: the shape parameter (α) and the scale parameter (β). The maximum likelihood estimators (MLEs) for the parameters of α and β were initially proposed by Birnbaum and Saunders [3], and their asymptotic distribution was subsequently determined by Engelhardt et al. [8]. Later, it was revealed by Ng et al. [22] that bias is exhibited by the MLEs of α and β when subjected to Monte Carlo simulation. To address this bias, a straightforward bias reduction method was introduced. The generalized confidence interval for parameters such as α , mean, quantiles, and the reliability function of the BirSau distribution was developed by Wang [32]. Lu and Chang [18] used the bootstrap method to create intervals for prediction for future realizations of the BirSau distribution. Bayesian estimation was employed by Achcar [1] to approximate the marginal posterior distributions of α and β. Wang et al. [33] used inverse-gamma (IG) priors for α and β, and computed Bayesian estimates and confidence intervals (CIs) for these parameters.

The mean is widely used across various domains for evaluating statistical significance. Furthermore, it can be extended to the parameter of interest, the ratio between the means of two populations, allowing for comparing two quantitative datasets. The ratio between the means has been applied in several fields. For example, Ivashchenko et al. [12] utilized the ratio between the means of teaching and motor training in physical education classes for junior school students. In agricultural research, Qiao et al. [27] compared two popular estimators for the ratio of means in independent normal variables: both an arithmetic average and a weighted average. Friedrich et al. [9] explored the ratio of means for analyzing continuous outcomes in meta-analysis. Siqueira [28] utilized the ratio between comparing sample size formulas for 2×2 crossover designs. Furthermore, numerous studies have examined the CI estimation of the ratio between the means, including Maneerat et al. [20] conducted research on CIs for the ratio of means in the delta-lognormal distribution. Krishnamoorthy et al. [15] propose the modified likelihood ratio test for estimating the CIs of the ratio of means of two independent normal distributions. Jana et al. [13] developed interval estimation methods for the mean, difference, and ratio of means for zero-adjusted inverse Gaussian distributions. Gao and Tain [10] constructed CIs for the difference and ratio of two gamma means using generalized inference, MOVER, and parametric bootstrapping. Panichkitkosolkul [25] proposed CIs for the process capability index C_p based on CIs for variance under non-normality. Panahi [23] estimated the confidence interval for the generalized inverted exponential distribution. Jantakoon and Volodin [14] proposed CIs for the shape and scale parameters of the BirSau distribution. They utilized the generalized pivotal approach and the percentile bootstrap approach. Puggard et al. [26] suggested CIs for the ratio of coefficients of variation of the BirSau distribution. Guo et al. [11] presented interval estimations for process capability indices related to the BirSau distribution. They employed the bootstrap CI and generalized CI methods. As far as we know, there has not been any prior published research that deals with statistical inference using the ratio between the means of BirSau distributions. As a result, the goal of this work is to construct CIs for the ratio of means based on the generalized confidence interval (GCI), bootstrap confidence interval (BCI), percentile bootstrap confidence interval (PBCI), Bayesian credible interval (BayCrI), and the highest posterior density (HPD). To further demonstrate the effectiveness of the proposed methods, we applied them to datasets containing daily wind speed data collected in the Thai provinces of Samut Prakan and Rayong during the period of July to September 2023.

The structure of this study is as follows: Section 2 described the methods for estimating the CI for the ratio of the means of the BirSau distribution. The Monte Carlo simulation investigation's numerical calculations are shown in Section 3. The empirical application of the proposed CI estimate methods utilizing wind speed data is shown in Section 4. The conclusions are presented in Section 5.

2 Methods

Suppose that $X = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from BirSau distribution with shape parameter α and scale parameter β, denoted as $BirSau(\alpha, \beta)$. The probability density function (PDF) and the cumulative distribution function (CDF) of X are given by

$$f(x;\alpha,\beta) = \frac{1}{2\alpha\beta\sqrt{2\pi}} \times \left\{ \left(\frac{\beta}{x}\right)^{\frac{1}{2}} + \left(\frac{\beta}{x}\right)^{\frac{3}{2}} \right\} \exp\left[-\frac{1}{2\alpha^2} \left(\frac{x}{\beta} + \frac{\beta}{x} - 2 \right) \right], \tag{1}$$

and

$$F(x) = \Phi\left[\frac{1}{\alpha} \left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}} \right) \right], \tag{2}$$

with $x > 0,\ \alpha, \beta > 0$, where $\Phi(\cdot)$ is the CDF of a standard normal distribution.

Let $X_1 = (X_{11}, X_{12}, \ldots, X_{1n_1})$ be a random sample of size n_1 from BirSau distribution with α_1 and β_1, denoted as $X_1 \sim BirSau\,(\alpha_1, \beta_1)$; and let $X_2 = (X_{21}, X_{22}, \ldots, X_{2n_2})$ be a random sample of size n_2 from BirSau distribution with α_2 and β_2, denoted as $X_2 \sim BirSau\,(\alpha_2, \beta_2)$. Using important property $(1/\alpha_i)\left(\sqrt{X_{ij}/\beta_i} - \sqrt{\beta_i/X_{ij}}\right) \sim N(0,1)$, the expected value and variance of $X_i = (X_{i1}, X_{i2}, ..., X_{in_i}), i = 1, 2$ can be obtained as

$$E(X_i) = \beta_i\left(1 + \frac{1}{2}{\alpha_i}^2\right), \tag{3}$$

and

$$Var(X_i) = (\alpha_i\beta_i)^2\left(1 + \frac{5}{4}{\alpha_i}^2\right). \tag{4}$$

Therefore, the ratio of the means of BirSau distributions is given by

$$\begin{aligned} \omega =& \frac{E(X_1)}{E(X_2)} \\ =& \frac{\beta_1\left(1 + \frac{1}{2}{\alpha_1}^2\right)}{\beta_2\left(1 + \frac{1}{2}{\alpha_2}^2\right)}. \end{aligned} \tag{5}$$

2.1 GCI

Weerahandi [34] introduced the foundational concept of the generalized pivotal quantity (GPQ) for constructing CIs.

Suppose that $X_i = (X_{i1}, X_{i2}, ..., X_{in_i}), i = 1, 2$ be a random sample sizes n_i from BirSau distribution with α_i and β_i denoted as $BirSau(\alpha_i\beta_i)$. According to Sun [29] and Wang [32], the GPQ for β_i can be defined as:

$$G_{\beta_i} := G_{\beta_i}(\mathbf{x}_i; G_i)\begin{cases} \max(\beta_{i1}, \beta_{i2}), & \text{if} \quad G_i \le 0, \\ \min(\beta_{i1}, \beta_{i2}), & \text{if} \quad G_i > 0, \end{cases} \tag{6}$$

where $\mathbf{x}_i = (x_{i1}, x_{i2}, ...x_{in_i})$ are the observed values of X_i and G_i follow a t-distribution with $n_i - 1$ degrees of freedom (denoted as $G_i \sim t(n_i - 1)$). Using Equation (6), we determine that β_{i1} and β_{i2} represent two possible solutions for

$$H_i\beta_i^2 - 2I_i\beta + J_i = 0, \tag{7}$$

where $H_i = [(n_i - 1)B_i^2 - (1/n_i)D_iG_i^2]$, $I_i = \left[(n_i - 1)A_iB_i - (1 - A_iB_i){G_i}^2\right]$, $A_i = n_i^{-1}\sum_{i=1}^{n_i}\sqrt{X_i}$, $B_i = n_i^{-1}\sum_{i=1}^{n_i}1/\sqrt{X_i}$, $C_i = \sum_{i=1}^{n_i}\left(\sqrt{X_i} - A_i\right)^2$, $D_i = \sum_{i=1}^{n_i}\left(1/\sqrt{X_i} - C_i\right)^2$ and $G_i \sim t(n_i - 1)$. The GPQ for α_i was subsequently developed by Wang [32], and it is deduced as follows:

$$G_{\alpha_i} := G_{\alpha_i}(\mathbf{x}_i; \upsilon_i, G_i) = \left[\frac{s_{i2}G_{\beta_i}^2 - 2n_iG_{\beta_i} + s_{i1}}{G_{\beta_i}\upsilon_i}\right]^{1/2}, \tag{8}$$

where $s_{i1} = \sum_{i=1}^{n_i} x_i$, $s_{i2} = \sum_{i=1}^{n_i} 1/x_i$ and υ_i follow a Chi-squared distribution with n_i degree of freedom (denoted as $\upsilon_i \sim \chi^2_{(n_i)}$). By substituting G_{α_i} and G_{β_i} into Equation (5), the GPQ for ω becomes

$$G_\omega = \frac{G_{\beta_1}\left(1 + \frac{1}{2}G_{\alpha 1}^2\right)}{G_{\beta_2}\left(1 + \frac{1}{2}G_{\alpha_2}^2\right)}. \tag{9}$$

Therefore, the approximated $100(1 - \gamma)\%$ two-sided CI for ω based on GCI is

$$\begin{aligned} CI_{GCI.\omega} =& [L_{GCI.\omega}, U_{GCI.\omega}] \\ =& [G_\omega(\gamma/2), G_\omega(1 - \gamma/2)], \end{aligned} \tag{10}$$

where $G_\omega(\gamma)$ denotes the $100\gamma\%$ percentile of G_ω.

Algorithm 1 : GCI

1. Simulate datasets $x_i = (x_{i1}, x_{i2}, \cdot, x_{in_i}), i = 1, 2$.
2. Calculate $A_i, B_i, C_i, D_i, s_{i1}$, and s_{i2} individually.
3. During the m^{th} iteration:
   (a) Generate $G_i \sim t(n_i - 1)$ and then compute G_{β_i} in Equation (6). If $G_{\beta_i} < 0$, regenerate $G_i \sim t(n_i - 1)$ again.
   (b) Generate random variables υ_i following a chi-squared distribution with degrees of freedom n_i, then determine G_{α_i} utilizing Equation (8).
   (c) To calculate the value of G_ω in Equation (9).
4. Perform step (3) iteratively for a total of M repetitions.
5. Calculate $G_\omega(\gamma/2)$ and $G_{1-\omega}(\gamma/2)$.

2.2 BCI

The bootstrap method, originally developed by Efron [6], is employed to mitigate bias associated with MLEs. Efron and Tibshirani [7] introduced the fundamental principles of bootstrapping. Researchers conduct calculations for bootstrap estimators of the α and β following the methodology outlined by Lemonte et al. [17]. According to their research, the most efficient approach for reducing bias was the constant-bias-correcting (CBC) parametric bootstrap method, which was suggested by MacKinnon and Smith [19]. This approach to bias correction in maximum-likelihood estimators entails harnessing the power of the bootstrap to manage the parameters defining the distribution. Thus, researchers can execute the process of applying the CBC parametric bootstrap to create a CI for the ratio of means as follows.

Consider a random sample, denoted as $\mathbf{x}_i = (x_{i1}, x_{i2}, ...x_{in_i})^T$ drawn from $BirSau(\alpha_i, \beta_i)$, where n_i represents the sample size. The MLEs for α_i and β_i, denoted as $\hat{\alpha}_i$ and $\hat{\beta}_i$ respectively, are determined by maximizing the log-likelihood functions specific to each parameter. This maximization is achieved using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton nonlinear optimization algorithm. As a result, a bootstrap sample of size n_i is derived from $BirSau(\hat{\alpha}_i, \hat{\beta}_i)$, denoted as $\mathbf{x}_i^* = (x_{i1}^*, x_{i2}^*, ..., x_{in}^*)^T$. From this sample, the corresponding bootstrapped values for $\hat{\alpha}_i$ and $\hat{\beta}_i$ are calculated and represented as $\hat{\alpha}_i^*$ and $\hat{\beta}_i^*$, respectively. This calculation is accomplished using the BFGS quasi-Newton nonlinear optimization algorithm. Consequently, $\hat{\alpha}_i^*$ and $\hat{\beta}_i^*$ based on b bootstrap samples can be obtained. The accurate estimations of $\hat{\alpha}_i^*$ and $\hat{\beta}_i^*$ are as follows, following the method proposed by MacKinnon and Smith [19]:

$$\tilde{\alpha}_{ik} = \hat{\alpha}_{ik}^* - 2\hat{b}\left(\hat{\alpha}_i, \alpha_i\right), \quad k = 1, 2, ..., b, \tag{11}$$

and

$$\tilde{\beta}_{ik} = \hat{\beta}_{ik}^* - 2\hat{b}\left(\hat{\beta}_i, \beta_i\right). \tag{12}$$

The bootstrap bias estimates of $\hat{\alpha}_i$ and $\hat{\beta}_i$ are:

$$\hat{b}\left(\hat{\alpha}_i, \alpha_i\right) = \hat{\alpha}_i^{*(\cdot)} - \hat{\alpha}_i \quad \text{and} \quad \hat{b}(\hat{\beta}_i, \beta_i) = \hat{\beta}_i^{*(\cdot)} - \hat{\beta}_i, \tag{13}$$

where $\hat{\alpha}_i^{*(\cdot)} = (1/b)\sum_{k=1}^{b} \hat{\alpha}_{ik}^*$ and $\hat{\beta}_i^{*(\cdot)} = (1/b)\sum_{k=1}^{b} \hat{\beta}_{ik}^*$. The percentile bootstrap estimator of $\hat{\alpha}_i^*$ and $\hat{\beta}_i^*$ are

$$\tilde{\alpha}^*{}_{ik} = 2\hat{\alpha}_i - \tilde{\alpha}_{ik}, \tag{14}$$

and

$$\tilde{\beta}^*{}_{ik} = 2\hat{\beta}_i - \tilde{\beta}_{ik}. \tag{15}$$

Therefore, the bootstrap estimator of the ratio of means can be obtained as

$$\hat{\omega}_k = \frac{\tilde{\beta}_{1k}\left(1 + \frac{1}{2}\tilde{\alpha}_{1k}^2\right)}{\tilde{\beta}_{2k}\left(1 + \frac{1}{2}\tilde{\alpha}_{2k}^2\right)}, \tag{16}$$

and the percentile bootstrap estimator of the ratio of means can be obtained as

$$\hat{\omega}_k^* = \frac{\tilde{\beta}_{1k}^*\left(1 + \frac{1}{2}\tilde{\alpha}^*{}_{1k}{}^2\right)}{\tilde{\beta}_{2k}^*\left(1 + \frac{1}{2}\tilde{\alpha}^*{}_{2k}{}^2\right)}. \tag{17}$$

Therefore, the approximated $100(1-\gamma)\%$ two-sided CI for ω based on BCI is

$$\begin{aligned} CI_{BCI.\omega} =& [L_{BCI.\omega}, U_{BCI.\omega}] \\ =& \left[\hat{\omega}\left(\gamma/2\right), \hat{\omega}\left(1-\gamma/2\right)\right], \end{aligned} \tag{18}$$

and the approximated $100(1-\gamma)\%$ two-sided CI for ω based on PBCI is

$$\begin{aligned} CI_{PBCI.\omega} =& [L_{PBCI.\omega}, U_{PBCI.\omega}] \\ =& \left[\hat{\omega}^*\left(\gamma/2\right), \hat{\omega}^*\left(1-\gamma/2\right)\right], \end{aligned} \tag{19}$$

where $\hat{\omega}\left(\gamma\right)$ and $\hat{\omega}^*\left(\gamma\right)$ denotes the $100\gamma\%$ percentile of $\hat{\omega}$ and $\hat{\omega}^*$, respectively.

Algorithm 2 : BCI

1. Simulate datasets $x_i = (x_{i1}, x_{i2}, \cdot, x_{in_i}), i = 1, 2$.
2. During the b^{th} iteration:
   (a) Sample x_i^* with replacement from x_i.
   (b) Calculate $\tilde{\alpha}_{ik}$ and $\tilde{\beta}_{ik}$ by using Equations (11) and (12).
   (c) Calculate $\tilde{\alpha}^*{}_{ik}$ and $\tilde{\beta}^*{}_{ik}$ by using Equations (14) and (15).
   (d) Calculate $\hat{\omega}_k$ and $\hat{\omega}_k^*$ by using Equations (16) and (17).
3. Execute step (2) iteratively for a total of B iterations.
4. For BCI, calculate $\hat{\omega}(\gamma/2)$ and $\hat{\omega}(1-\gamma/2)$.
5. For PBCI, calculate $\hat{\omega}^*(\gamma/2)$ and $\hat{\omega}^*(1-\gamma/2)$.

2.3 BayCrI

As the use of noninformative priors directly results in an improper posterior distribution for the BirSau distribution, and continuous conjugate priors do not exist, Wang et al. [33] opted to employ an appropriate prior with known hyperparameters to ensure that the posteriors remain proper. In this approach, β_i and α_i^2 are assumed to follow an IG distribution, represented as

$IG\left(\beta_i\,|a_{i1},b_{i1}\right)$ and $IG\left(\alpha_i^2\,|a_{i2},b_{i2}\right)$. The PDF of the IG distribution is defined as follows: if η is given an IG distribution $IG(\eta\,|a,b)$ with parameters a and b,

$$\pi\left(\eta\,|a,b\right)=\frac{b^a}{\Gamma\left(a\right)}\eta^{-a-1}\exp\left(-\frac{b}{\eta}\right),\quad a,b>0. \tag{20}$$

Let $\mathbf{X}_i=\left(X_{i1},X_{i2},...X_{in_i}\right)^T$, $i=1,2$ be a random sample from BirSau distributions with α_i and β_i. The observations of $\mathbf{X}$ are $\mathbf{x}_i=\left(x_{i1},x_{i2},...,x_{in}\right)^T$. Therefore, the likelihood function without including the constant additive term, can be expressed as:

$$L\left(\mathbf{x}_i\,|\alpha_i,\beta_i\right)\propto\frac{1}{\alpha_i^{n_i}\beta_i^{n_i}}\prod_{k=1}^{n_i}\left[\left(\frac{\beta_i}{x_{ik}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ik}}\right)^{\frac{3}{2}}\right]\exp\left[-\sum_{k=1}^{n_i}\frac{1}{2\alpha_i^2}\left(\frac{x_{ik}}{\beta_i}+\frac{\beta_i}{x_{ik}}-2\right)\right]. \tag{21}$$

By combining the likelihood function with the IG priors for β_i and α_i^2, the joint posterior density function of $\left(\beta_i,\alpha_i^2\right)$ becomes

$$\begin{aligned}f(\alpha_i^2,\beta_i\,|\mathbf{x}_i)\propto&L\left(\mathbf{x}_i\,|\alpha_i,\beta_i\right)\pi\left(\beta_i\,|a_{i1},b_{i1}\right)\pi\left(\alpha^2\,|a_{i2},b_{i2}\right)\\\propto&\frac{1}{\left(\alpha_i^2\right)^{\frac{n_i}{2}}\beta^{n_i}}\prod_{k=1}^{n_i}\left[\left(\frac{\beta_i}{x_{ik}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ik}}\right)^{\frac{3}{2}}\right]\exp\left[-\sum_{k=1}^{n_i}\frac{1}{2\alpha_i^2}\left(\frac{x_{ik}}{\beta_i}+\frac{\beta_i}{x_{ik}}-2\right)\right]\\&\times\beta_i^{-a_{i1}-1}\exp\left(-\frac{b_{i1}}{\beta_i}\right)\left(\alpha_i^2\right)^{-a_{i2}-1}\exp\left(-\frac{b_{i2}}{\alpha_i^2}\right).\end{aligned} \tag{22}$$

The marginal posterior distribution of β_i given the data is as follows:

$$\begin{aligned}\pi\left(\beta_i\,|\mathbf{x}_i\right)\propto&\beta_i^{-(n_i+a_{i1}+1)}\exp\left(-\frac{b_{i1}}{\beta_i}\right)\prod_{k=1}^{n_i}\left[\left(\frac{\beta_i}{x_{ik}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ik}}\right)^{\frac{3}{2}}\right]\\&\times\left[\sum_{k=1}^{n_i}\frac{1}{2}\left(\frac{x_{ik}}{\beta_i}+\frac{\beta}{x_{ik}}-2\right)+b_{i2}\right]^{-(n_i+1)/2-a_{i2}}.\end{aligned} \tag{23}$$

The conditional posterior distribution of α_i^2 given β_i and the data is

$$\pi\left(\alpha_i^2\,|\mathbf{x}_i,\beta_i\right)\propto IG\left(\frac{n_i}{2}+a_{i2},\frac{1}{2}\sum_{k=1}^{n_i}\left(\frac{x_{ik}}{\beta_i}+\frac{\beta_i}{x_{ik}}-2\right)+b_{i2}\right). \tag{24}$$

The samples from Equations (23) and (26) are obtained using Markov Chain Monte Carlo techniques. The posterior samples of β_i were produced by Wang et al. [33] using the generalized ratio-of-uniforms approach described in the next part because the marginal distribution (23) is analytically intractable. Meanwhile, the R software suite's *LearnBayes* package makes it simple to get the posterior samples of α_i^2. Thus, by calculating the square root of α_i^2, we may obtain α_i.

2.3.1 Generalized ratio-of-uniforms method

Wakefield et al. [31] presented the generalized ratio-of-uniforms approach for establishing an effective sampling technique for posterior simulation based on Equation (23). A summary of the algorithm is presented as follows: Suppose that a pair of random variables (u_i,v_i) are uniformly distributed over the region

$$K\left(r_i\right)=\left\{(u_i,v_i):0<u_i\leq\left[\pi\left(\frac{v_i}{{u_i}^{r_i}}\,|\mathbf{x}_i\right)\right]^{1/(r_i+1)}\right\},\quad r_i\geq0, \tag{25}$$

where r_i is a constant and $\pi\left(\cdot\,|\mathbf{x}_i\right)$ is given by using Equation (23). Then, $\beta_i = v_i/u_i^{r_i}$ has the density of the $\pi\left(\beta_i|\mathbf{x}_i\right)/\int \pi\left(\beta_i|\mathbf{x}_i\right) d\beta_i$.

To generate random samples uniformly distributed within $K\left(r_i\right)$, Wang et al. [33] employed the acceptreject method using a suitable one-dimensional enveloping rectangle, denoted as $[0, a\left(r_i\right)] \times [b^-(r_i), b^+(r_i)]$ where

$$a(r_i) = \sup_{\beta_i>0} \left\{ \left[\pi(\beta_i|\mathbf{x}_i)^{1/(r_i+1)} \right] \right\}, \tag{26}$$

$$b^-\left(r_i\right) = \inf_{\beta>0} \left\{ \beta_i [\pi\left(\beta_i|\mathbf{x}_i\right)]^{r_i/(r_i+1)} \right\}, \tag{27}$$

and

$$b^+\left(r_i\right) = \sup_{\beta_i>0} \left\{ \beta_i [\pi\left(\beta_i|\mathbf{x}_i\right)]^{r_i/(r_i+1)} \right\}. \tag{28}$$

Wang et al. [33] demonstrated that both $a\left(r_i\right)$ and $b^+\left(r_i\right)$ are finite, while $b^-\left(r_i\right)$ equals zero. Following this, a candidate value for the variable $\beta_i = v_i/u_i^{r_i}$ is accepted if it satisfies the condition $u_i \leq [\pi\left(\beta_i\,|\mathbf{x}\right)]^{1/(r_i+1)}$; if not, repeat the entire process. Consequently, the steps below can be used to find the Bayesian CI for the ratio of means of the BirSau distribution:

Algorithm 3 : BayCrI

1. Indicate the $a_{i1}, a_{i2}, b_{i1}, b_{i2}$ and r_i values, when $i = 1, 2$.
2. Compute the $a\left(r_i\right)$ and $b^+\left(r_i\right)$ by using Equation (26) and (28).
3. For the interaction:
   (a) Generate u_i from $Unif(0, a\left(r_i\right))$, where $Unif(0, a\left(r_i\right))$ has an uniform distribution with parameter 0 and $a\left(r_i\right)$.
   (b) Generate v_i from $Unif(0, b^+\left(r_i\right))$.
   (c) Calculate ρ_i by using $\rho_i = v_i/u_i^{r_i}$. In the case where $u_i \leq [\pi\left(\rho_i\,|\mathbf{x}_i\right)]^{1/(r_i+1)}$, set $\beta_{(ik)} = \rho_i$; if not, repeat steps (a) and (b).
   (d) Generate λ_i based on $IG\left(\frac{n_i}{2} + a_{i2}, \frac{1}{2}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\beta_{(ik)}} + \frac{\beta_{(ik)}}{x_{ij}} - 2\right) + b_{i2}\right)$, then $\alpha_{(ik)} = \sqrt{\lambda_i}$.
4. Calculate the Bayesian estimator for the ratio between means (denoted as ω'), using the following

$$\omega'_{(k)} = \frac{\left(\beta_{(1k)}\right)\left(1 + \frac{1}{2}\alpha_{(1k)}{}^2\right)}{\left(\beta_{(2k)}\right)\left(1 + \frac{1}{2}\alpha_{(2k)}{}^2\right)}. \tag{29}$$

5. Perform steps (3) and (4) repeatedly M times.
6. Compute the approximated $100\left(1-\gamma\right)\%$ two-sided CI for ω based on BayCrI is

$$\begin{aligned} CI_{BayCrI.\omega} &= [L_{BayCrI.\omega}, U_{BayCrI.\omega}] \\ &= \left[\omega'\left(\gamma/2\right), \omega'\left(1-\gamma/2\right)\right], \end{aligned} \tag{30}$$

   where $\omega'\left(\gamma\right)$ denotes the $100\gamma\%$ percentile of ω'.

In this study, after calculating the Bayesian estimator for the ratio of means in step (5), we utilized the *hdi* function from the *HDInterval* package in the R software suite to generate the Highest Posterior Density (HPD) interval for the ratio of means.

3 Simulation studies

The study utilizes the R statistical software to conduct a Monte Carlo simulation that investigates five methods (GCI, BCI, PBCI, BayCrI, and HPD) for constructing new confidence intervals (CIs) for the ratio between the means of two BirSau distributions. The effectiveness of these five methods is evaluated by analyzing their coverage probabilities (CPs) and average widths (AWs). The best-performing method is determined by achieving CP values that are at least close to the nominal confidence level of 0.95. Additionally, the method with the shortest AW is considered the most effective. The simulation settings include 5,000 replications, 5,000 pivotal quantities for GCI, $b = 500$ for BCI and PBCI, and $M = 1,000$ for BayCrI and HPD interval. The scale parameter values β_1 and β_2 are fixed at 1 for all cases. Following the recommendation of Wang et al. [33], hyperparameters $r_1 = r_2 = 2$ and $a_{11} = a_{12} = a_{21} = a_{22} = b_{11} = b_{12} = b_{21} = b_{22} = 10^{-4}$ are adopted for the BayCrI and HPD methods.

For the ratio between the means of BirSau distributions, the data were generated from two independent BirSau distributions, $X_1 \sim BirSau(\alpha_1, \beta_1)$ and $X_2 \sim BirSau(\alpha_2, \beta_2)$. For equal sample sizes $(n_1 = n_2)$, we used (10,10), (20,20), (30,30), (50,50), or (100,100) and for unequal sample sizes $(n_1 \neq n_2)$, we used (10,20), (20,30), (30,50), or (50,100). For shape parameters, we set (α_1, α_2) = (0.25,0.25), (0.25,0.50), (0.25,0.75), (0.25,1.00), (0.50,0.50), (0.50,1.00), (0.75,0.75), (0.75,1.00), or (1.00,1.00). The CIs for the ratio between the means, with equal and unequal sample sizes, are listed in Tables 1 and 2, respectively. The CP and AWs of the 95% CIs are provided in the tables.

Firstly, the performances of CIs for the ratio between the means of BirSau distribution with equal sample sizes. It can be seen that the CPs and AWs in small (n_1, n_2) = (10,10), (20,20), (30,30), and medium sample sizes (n_1, n_2) = (50,50). It was observed that the GCI, BayCrI, and HPD methods yielded CP values at least or close to the nominal confidence level of 0.95. However, the CPs of the BCI and PBCI methods were less than the nominal confidence level of 0.95. When dealing with large sample sizes (n_1, n_2) = (100,100), all five methods, namely GCI, BCI, PBCI, BayCrI, and HPD, exhibited CPs at least or close to the nominal confidence level of 0.95. However, upon considering the AWs, we observed that the BCI and PBCI methods have similar AW values, providing the shortest AW compared to the GCI, BayCrI, and HPD methods. However, in the scenarios involving small and medium sample sizes, it was discovered that the CPs of the BCI and PBCI methods were less than the nominal confidence level of 0.95, so they were not considered. Hence, the AWs of the GCI, Bay, and HPD methods were taken into account. The analysis revealed that the HPD method yielded the lowest AW value. Conversely, for large groups, all five methods exhibited CPs at least or close to the nominal confidence level of 0.95. It was observed that the BCI method produced the shortest AW as shown in Figure 1.

Secondly, we investigated the CPs and AWs of the proposed CIs for the ratio between the means of the BirSau distribution, considering unequal sample sizes. Analyzing small sample sizes (n_1, n_2) = (10,20), (20,30), (30,50), revealed that the GCI, BayCrI, and HPD methods achieved CP values at least or close to the nominal confidence level of 0.95. For larger sample sizes (n_1, m_2) = (50,100), all five methods demonstrated CP values at least or close to the nominal confidence level of 0.95. However, while evaluating AWs for small sample sizes, it was found that the BCI and PBCI methods provided the shortest AWs. Despite this, it was noted that the CPs of the BCI and PBCI methods were less than the nominal confidence level of 0.95, leading to their exclusion from consideration. We consistently observed that the HPD method consistently yielded the shortest AWs. Conversely, in the case of larger sample sizes, the BCI method exhibited the shortest AWs, as demonstrated in Figure 2.

Furthermore, when the shape parameters were set to (α_1, α_2) = (0.5, 0.5), (0.5, 0.75), (0.5, 1), (0.75, 0.75), (0.75, 1), or (1, 1), it was observed that the AWs produced by the BCI method were shorter than those generated by the PBCI method. Finally, when we made changes to certain parameters, such as maintaining α_1 at a constant value while increasing α_2, we observed that the AWs increased. However, when we examined the impact of alterations in sample size (n_1, n_2), we discovered that increasing the sample size led to shorter AWs.

Table 1: The CPs and AWs of the 95% CIs for the ratio between the means of BirSau distributions, where the sample sizes are equal $(n_1 = n_2)$.

(n_1, n_2)	(α_1, α_2)	**CP (AW)**				
		GCI	BCI	PBCI	BayCrI	HPD
(10,10)	(0.25,0.25)	**0.9612**	0.9178	0.9268	**0.9470**	**0.9496**
		(0.5221)	(0.4197)	(0.4255)	(0.4873)	**(0.4783)**
	(0.25,0.50)	**0.9532**	0.9130	0.9110	0.9412	**0.9448**
		(1.0326)	(0.7485)	(0.7372)	(0.9567)	**(0.8927)**
	(0.25,0.75)	**0.9562**	0.9122	0.8946	0.9434	**0.9478**
		(1.9630)	(1.2428)	(1.1959)	(1.7779)	**(1.5584)**
	(0.25,1.00)	**0.9490**	0.9074	0.8736	0.9358	**0.9458**
		(3.5550)	(1.9196)	(1.8022)	(3.0383)	**(2.5105)**
	(0.50,0.50)	**0.9642**	0.9284	0.9436	**0.9500**	**0.9492**
		(1.1496)	(0.8885)	(0.9367)	(1.0707)	**(1.0179)**
	(0.50,0.75)	**0.9628**	0.9280	0.9370	**0.9492**	**0.9560**
		(2.0026)	(1.3464)	(1.3711)	(1.8180)	**(1.6247)**
	(0.50,1.00)	**0.9564**	0.9210	0.9194	0.9422	**0.9478**
		(3.4310)	(1.9664)	(1.9393)	(2.9566)	**(2.4890)**
	(0.75,0.75)	**0.9628**	0.9336	**0.9624**	**0.9544**	**0.9534**
		(1.9797)	(1.4713)	(1.6555)	(1.8140)	**(1.6433)**
	(0.75,1.00)	**0.9536**	0.9300	**0.9512**	**0.9452**	**0.9456**
		(3.3108)	(2.0547)	**(2.2205)**	(2.8577)	(2.4362)
	(1.00,1.00)	**0.9578**	0.9434	**0.9676**	**0.9480**	**0.9490**
		(3.1548)	(2.1934)	(2.5685)	(2.7420)	**(2.3457)**
(20,20)	(0.25,0.25)	**0.9582**	0.9356	0.9388	**0.9532**	**0.9490**
		(0.3347)	(0.3025)	(0.3044)	(0.3244)	**(0.3205)**
	(0.25,0.50)	0.9444	0.9234	0.9260	0.9390	0.9402
		(0.6138)	(0.5326)	(0.5281)	(0.5936)	(0.5744)
	(0.25,0.75)	**0.9500**	0.9320	0.9176	0.9444	**0.9476**
		(1.0677)	(0.8767)	(0.8610)	(1.0231)	**(0.9636)**
	(0.25,1.00)	**0.9470**	0.9242	0.9042	0.9398	0.9438
		(1.7243)	(1.3278)	(1.3022)	(1.6198)	(1.4842)
	(0.50,0.50)	**0.9542**	0.9362	0.9444	**0.9468**	**0.9476**
		(0.7068)	(0.6308)	(0.6462)	(0.6842)	**(0.6677)**
	(0.50,0.75)	**0.9538**	0.9374	**0.9454**	**0.9478**	**0.9486**
		(1.1096)	(0.9343)	**(0.9436)**	(1.0654)	(1.0120)
	(0.75,0.75)	**0.9480**	0.9336	**0.9546**	0.9408	0.9434
		(1.1344)	(0.9950)	**(1.0500)**	(1.0884)	(1.0409)
	(0.75,1.00)	**0.9550**	0.9410	**0.9564**	**0.9506**	**0.9508**
		(1.6683)	(1.3689)	**(1.4241)**	(1.5781)	(1.4686)
	(1.00,1.00)	**0.9508**	0.9420	**0.9648**	**0.9470**	**0.9464**
		(1.5916)	(1.3709)	(1.4991)	(1.4996)	**(1.4019)**

Table 1: The CPs and AWs of the 95% CIs for the ratio between the means of BirSau distributions, where the sample sizes are equal ($n_1 = n_2$) (continued).

(n_1, n_2)	(α_1, α_2)	CP (AW)				
		GCI	BCI	PBCI	BayCrI	HPD
(30,30)	(0.25,0.25)	**0.9522**	0.9378	0.9406	**0.9476**	0.9424
		(0.2662)	(0.2486)	(0.2499)	**(0.2606)**	(0.2577)
	(0.25,0.50)	**0.9486**	0.9322	0.9336	0.9416	0.9430
		(0.4799)	(0.4370)	(0.4344)	(0.4691)	(0.4583)
	(0.25,0.75)	0.9400	0.9302	0.9240	0.9382	0.9386
		(0.8129)	(0.7125)	(0.7049)	(0.7894)	(0.7574)
	(0.25,1.00)	**0.9504**	0.9396	0.9304	**0.9464**	**0.9502**
		(1.2908)	(1.0842)	(1.0731)	(1.2367)	**(1.1652)**
	(0.50,0.50)	**0.9500**	0.9400	**0.9448**	0.9444	**0.9462**
		(0.5546)	(0.5134)	**(0.5218)**	(0.5423)	(0.5324)
	(0.50,0.75)	**0.9494**	0.9346	0.9392	0.9432	0.9404
		(0.8416)	(0.7517)	(0.7567)	(0.8175)	(0.7894)
	(0.50,1.00)	**0.9466**	0.9352	0.9362	0.9402	0.9420
		(1.2721)	(1.0864)	(1.0874)	(1.2172)	(1.1531)
	(0.75,0.75)	**0.9520**	0.9390	**0.9518**	0.9444	0.9442
		(0.8643)	(0.7926)	**(0.8187)**	(0.8401)	(0.8147)
	(0.75,1.00)	**0.9550**	**0.9462**	**0.9550**	**0.9490**	**0.9484**
		(1.2360)	**(1.0842)**	(1.1104)	(1.1862)	(1.1309)
	(1.00,1.00)	**0.9534**	**0.9486**	**0.9632**	**0.9518**	**0.9498**
		(1.1958)	**(1.0826)**	(1.1404)	(1.1451)	(1.0942)
(50,50)	(0.25,0.25)	**0.9484**	0.9406	0.9370	**0.9460**	0.9422
		(0.2023)	(0.1936)	(0.1943)	**(0.1995)**	(0.1976)
	(0.25,0.50)	**0.9534**	**0.9466**	**0.9468**	**0.9524**	**0.9506**
		(0.3598)	(0.3393)	**(0.3381)**	(0.3542)	(0.3484)
	(0.25,0.75)	**0.9528**	0.9444	0.9366	**0.9508**	**0.9508**
		(0.5985)	(0.5521)	(0.5475)	(0.5858)	**(0.5703)**
	(0.25,1.00)	**0.9496**	0.9396	0.9340	**0.9476**	**0.9468**
		(0.9320)	(0.8329)	(0.8264)	(0.9000)	**(0.8664)**
	(0.50,0.50)	**0.9542**	**0.9446**	**0.9498**	**0.9508**	**0.9498**
		(0.4164)	**(0.3971)**	(0.4001)	(0.4100)	(0.4043)
	(0.50,0.75)	**0.9516**	0.9418	0.9408	**0.9496**	**0.9450**
		(0.6208)	(0.5782)	(0.5802)	(0.6083)	**(0.5942)**
	(0.50,1.00)	**0.9544**	0.9428	0.9410	**0.9488**	**0.9466**
		(0.9214)	(0.8348)	(0.8351)	(0.8936)	**(0.8627)**
	(0.75,0.75)	**0.9554**	**0.9490**	**0.9544**	**0.9532**	**0.9504**
		(0.6408)	**(0.6061)**	(0.6171)	(0.6270)	(0.6143)
	(0.75,1.00)	**0.9484**	0.9422	**0.9478**	**0.9468**	**0.9452**
		(0.9115)	(0.8375)	**(0.8486)**	(0.8823)	(0.8553)
	(1.00,1.00)	**0.9512**	**0.9510**	**0.9526**	**0.9472**	**0.9460**
		(0.8686)	**(0.8120)**	(0.8364)	(0.8388)	(0.8148)

Table 1: The CPs and AWs of the 95% CIs for the ratio between the means of BirSau distributions, where the sample sizes are equal ($n_1 = n_2$) (continued).

(n_1, n_2)	(α_1, α_2)	CP (AW)				
		GCI	BCI	PBCI	BayCrI	HPD
(100 ,100)	(0.25,0.25)	**0.9516**	0.9444	**0.9468**	**0.9506**	**0.9478**
		(0.1410)	(0.1376)	**(0.1376)**	(0.1397)	(0.1385)
	(0.25,0.50)	**0.9462**	0.9394	0.9398	0.9442	0.9386
		(0.2484)	(0.2402)	(0.2399)	(0.2458)	(0.2428)
	(0.25,0.75)	**0.9508**	**0.9446**	0.9392	**0.9478**	**0.9456**
		(0.4079)	**(0.3889)**	(0.3873)	(0.4014)	(0.3946)
	(0.25,1.00)	**0.9568**	**0.9452**	0.9400	**0.9530**	**0.9498**
		(0.6237)	(0.5832)	(0.5807)	(0.6061)	(0.5927)
	(0.50,0.50)	**0.9510**	**0.9464**	0.9434	**0.9480**	**0.9466**
		(0.2884)	**(0.2815)**	(0.2804)	(0.2852)	(0.2821)
	(0.50,0.75)	**0.9530**	**0.9478**	**0.9490**	**0.9498**	**0.9470**
		(0.4248)	**(0.4071)**	(0.4083)	(0.4187)	(0.4122)
	(0.50,1.00)	**0.9576**	**0.9512**	**0.9528**	**0.9530**	**0.9510**
		(0.6213)	**(0.5857)**	**(0.5857)**	(0.6067)	(0.5941)
	(0.75,0.75)	**0.9458**	0.9428	**0.9450**	0.9438	0.9430
		(0.4402)	(0.4247)	**(0.4290)**	(0.4334)	(0.4273)
	(0.75,1.00)	**0.9508**	**0.9460**	**0.9512**	**0.9468**	**0.9482**
		(0.6126)	**(0.5818)**	(0.5853)	(0.5980)	(0.5864)
	(1.00,1.00)	**0.9522**	**0.9498**	**0.9536**	**0.9516**	**0.9482**
		(0.5880)	**(0.5621)**	(0.5694)	(0.5722)	(0.5618)

Note: Bold denotes the best-performing method.

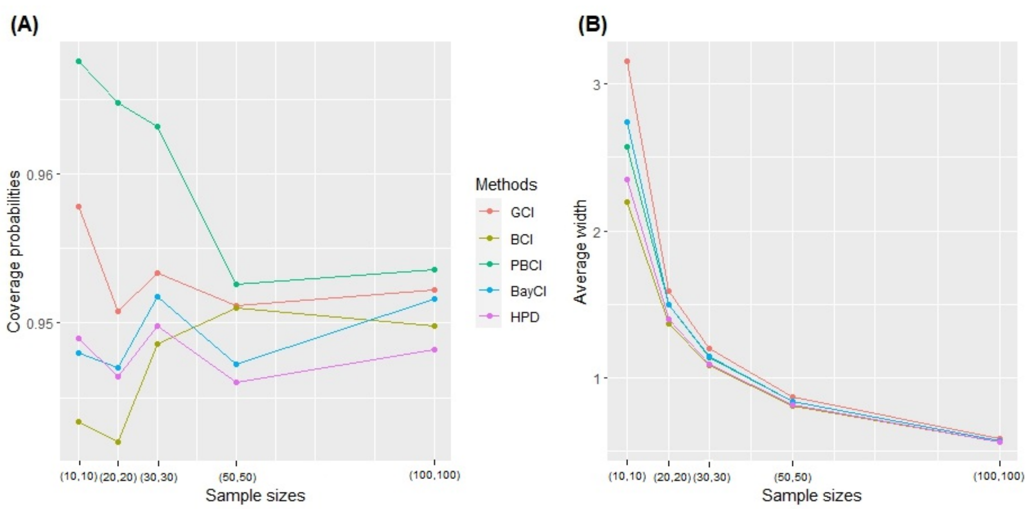

Figure 1: A summary of the CPs and AWs of the methods for equal sample sizes at $(\alpha_1, \alpha_2) = (1, 1)$.

Table 2: The CPs and AWs of the 95% CIs for the ratio between the means of BirSau distributions, where the sample sizes are unequal ($n_1 \neq n_2$).

(n_1, n_2)	(α_1, α_2)	**CP (AW)**				
		GCI	BCI	PBCI	BayCrI	HPD
(10,20)	(0.25,0.25)	**0.9622**	0.9262	0.9302	**0.9512**	**0.9462**
		(0.4263)	(0.3629)	(0.3698)	(0.4034)	**(0.3991)**
	(0.25,0.50)	**0.9594**	0.9328	0.9322	**0.9506**	**0.9458**
		(0.6850)	(0.5804)	(0.5830)	(0.6552)	**(0.6376)**
	(0.25,0.75)	**0.9528**	0.9244	0.9212	**0.9456**	0.9436
		(1.1272)	(0.9177)	(0.9090)	**(1.0751)**	(1.0173)
	(0.25,1.00)	**0.9500**	0.9332	0.9216	0.9442	**0.9486**
		(1.7848)	(1.3706)	(1.3524)	(1.6718)	**(1.5390)**
	(0.50,0.50)	**0.9546**	0.9246	0.9350	0.9414	0.9430
		(0.8924)	(0.7737)	(0.8364)	(0.8484)	(0.8330)
	(0.50,0.75)	**0.9530**	0.9290	**0.9460**	0.9420	0.9434
		(1.2819)	(1.0734)	**(1.1316)**	(1.2168)	(1.1694)
	(0.50,1.00)	**0.9532**	0.9342	0.9432	**0.9470**	**0.9482**
		(1.8939)	(1.4992)	(1.5543)	(1.7701)	**(1.6545)**
	(0.75,0.75)	**0.9572**	0.9316	0.9438	**0.9462**	**0.9480**
		(1.3917)	(1.2454)	(1.4587)	(1.3151)	**(1.2689)**
	(0.75,1.00)	**0.9564**	0.9412	**0.9616**	**0.9468**	0.9410
		(1.9590)	(1.6644)	(1.9080)	**(1.8373)**	(1.7290)
	(1.00,1.00)	**0.9582**	0.9358	**0.9504**	**0.9492**	**0.9454**
		(1.9769)	(1.8601)	(2.3177)	(1.8468)	**(1.7385)**
(20,30)	(0.25,0.25)	**0.9566**	0.9394	0.9418	**0.9498**	**0.9464**
		(0.3007)	(0.2764)	(0.2788)	(0.2924)	**(0.2895)**
	(0.25,0.50)	**0.9576**	0.9416	0.9394	**0.9534**	**0.9482**
		(0.5051)	(0.4581)	(0.4567)	(0.4929)	**(0.4821)**
	(0.25,0.75)	**0.9500**	0.9332	0.9274	**0.9448**	0.9424
		(0.8286)	(0.7260)	(0.7202)	**(0.8032)**	(0.7721)
	(0.25,1.00)	**0.9462**	0.9308	0.9174	0.9406	0.9408
		(1.2992)	(1.0891)	(1.0810)	(1.2388)	(1.1691)
	(0.50,0.50)	**0.9532**	0.9388	**0.9490**	**0.9486**	**0.9492**
		(0.6239)	(0.5742)	**(0.5922)**	(0.6066)	(0.5971)
	(0.50,0.75)	**0.9566**	**0.9446**	**0.9498**	**0.9524**	**0.9500**
		(0.9107)	**(0.8139)**	(0.8295)	(0.8825)	(0.8548)
	(0.50,1.00)	**0.9512**	0.9384	0.9430	**0.9462**	0.9444
		(1.3371)	(1.1481)	(1.1620)	**(1.2800)**	(1.2170)
	(0.75,0.75)	**0.9534**	0.9420	**0.9548**	**0.9482**	0.9430
		(0.9649)	(0.8931)	(0.9523)	**(0.9336)**	(0.9081)
	(0.75,1.00)	**0.9464**	0.9366	**0.9516**	0.9408	0.9432
		(1.3576)	(1.2053)	**(1.2714)**	(1.2995)	(1.2430)
	(1.00,1.00)	**0.9562**	0.9440	**0.9542**	**0.9470**	**0.9520**
		(1.3420)	(1.2506)	(1.3877)	(1.2840)	**(1.2311)**

Table 2: The CPs and AWs of the 95% CIs for the ratio between the means of BirSau distributions, where the sample sizes are unequal $(n_1 \neq n_2)$ (continued).

(n_1, n_2)	(α_1, α_2)	CP (AW)				
		GCI	BCI	PBCI	BayCrI	HPD
(30,50)	(0.25,0.25)	**0.9536**	0.9386	0.9404	**0.9490**	0.9460
		(0.2361)	(0.2234)	(0.2248)	(0.2316)	**(0.2295)**
	(0.25,0.50)	**0.9472**	0.9316	0.9374	0.9430	0.9404
		(0.3847)	(0.3615)	(0.3614)	(0.3779)	(0.3719)
	(0.25,0.75)	**0.9550**	**0.9460**	0.9430	**0.9522**	**0.9526**
		(0.6200)	**(0.5707)**	(0.5689)	(0.6063)	(0.5905)
	(0.25,1.00)	**0.9460**	0.9352	0.9256	0.9426	0.9424
		(0.9456)	(0.8472)	(0.8421)	(0.9134)	(0.8798)
	(0.50,0.50)	**0.9516**	0.9390	**0.9462**	**0.9480**	0.9430
		(0.4833)	(0.4582)	**(0.4673)**	(0.4741)	(0.4681)
	(0.50,0.75)	**0.9512**	**0.9446**	**0.9500**	**0.9496**	**0.9494**
		(0.6871)	**(0.6402)**	(0.6488)	(0.6715)	(0.6574)
	(0.50,1.00)	**0.9484**	0.9422	**0.9460**	0.9440	**0.9462**
		(0.9835)	(0.8939)	**(0.9007)**	(0.9525)	(0.9222)
	(0.75,0.75)	**0.9506**	0.9406	**0.9474**	**0.9460**	**0.9458**
		(0.7452)	(0.7080)	(0.7397)	(0.7271)	**(0.7140)**
	(0.75,1.00)	**0.9508**	0.9410	**0.9522**	**0.9472**	**0.9448**
		(1.0035)	(0.9318)	(0.9664)	(0.9731)	**(0.9459)**
	(1.00,1.00)	**0.9476**	0.9400	**0.9490**	0.9408	0.9388
		(1.0002)	(0.9535)	(1.0197)	**(0.9645)**	(0.9393)
(50, 100)	(0.25,0.25)	**0.9498**	0.9398	0.9410	**0.9484**	**0.9446**
		(0.1743)	(0.1682)	(0.1690)	(0.1721)	**(0.1706)**
	(0.25,0.50)	**0.9522**	**0.9494**	0.9444	**0.9492**	**0.9486**
		(0.2723)	**(0.2623)**	(0.2625)	(0.2690)	(0.2658)
	(0.25,0.75)	**0.9550**	**0.9474**	**0.9464**	**0.9520**	**0.9502**
		(0.4273)	(0.4074)	**(0.4063)**	(0.4204)	(0.4133)
	(0.25,1.00)	**0.9460**	0.9352	0.9256	0.9426	0.9424
		(0.9456)	(0.8472)	(0.8421)	(0.9134)	(0.8798)
	(0.50,0.50)	**0.9452**	0.9406	0.9436	0.9434	0.9436
		(0.3553)	(0.3433)	(0.3486)	(0.3504)	(0.3469)
	(0.50,0.75)	**0.9512**	**0.9446**	**0.9500**	**0.9496**	**0.9494**
		(0.4884)	**(0.4686)**	(0.4732)	(0.4808)	(0.4740)
	(0.50,1.00)	**0.9496**	**0.9448**	**0.9486**	**0.9488**	**0.9452**
		(0.6808)	**(0.6427)**	(0.6467)	(0.6644)	(0.6517)
	(0.75,0.75)	**0.9480**	0.9388	0.9428	0.9438	0.9404
		(0.5404)	(0.5228)	(0.5368)	(0.5299)	(0.5230)
	(0.75,1.00)	**0.9516**	**0.9454**	**0.9534**	**0.9524**	**0.9464**
		(0.7186)	**(0.6860)**	(0.7012)	(0.7014)	(0.6893)
	(1.00,1.00)	**0.9502**	**0.9458**	**0.9500**	**0.9494**	**0.9464**
		(0.7203)	(0.6962)	(0.7244)	(0.6988)	**(0.6872)**

Note: Bold denotes the best-performing method.

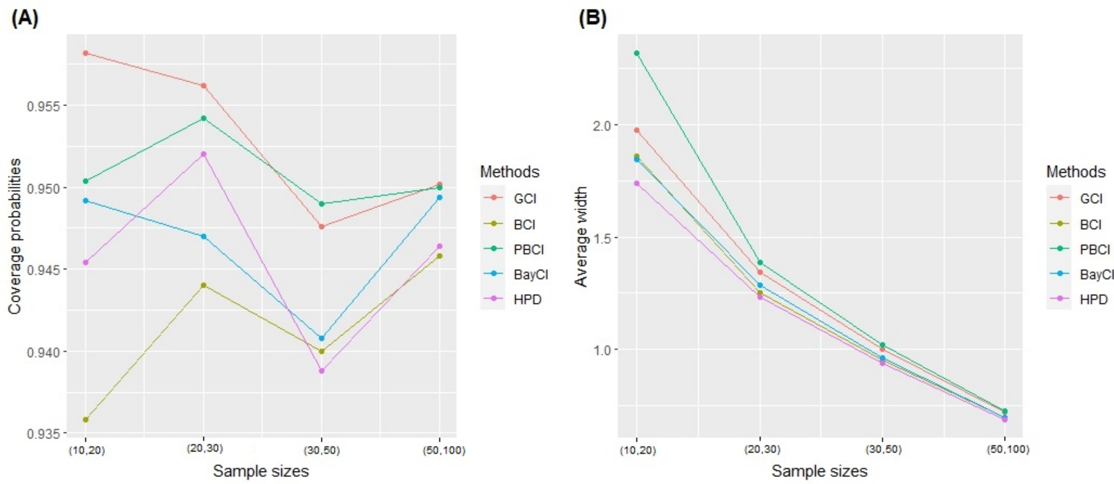

Figure 2: A summary of the CPs and AWs of the methods for unequal sample sizes at $(\alpha_1, \alpha_2) = (1, 1)$.

4 An empirical application

People living in industrial areas often face health problems, such as respiratory illnesses, attributed to dust accumulation. Amuaylojaroen et al. [2] study has revealed that there are factors influencing the distribution of dust levels, with one of them being wind speed, which can make dust particles lighter. Furthermore, studies by Mohammadi et al. [21] and Leiva et al. [16] have incorporated wind speed data into the BirSau distribution model. Consequently, we chose wind speed data from two separate industrial provinces in Thailand: Samut Prakan Province in the central region and Rayong Province in the eastern region.

We utilized the average daily wind speed datasets from Samut Prakan and Rayong province in Thailand to assess the effectiveness of our proposed methods for establishing the ratio between two means of BirSau distributions (as detailed in Table 3). These datasets were collected between July and September 2023 by the Thai Meteorological Department [30]. Since the wind speed data comprises exclusively positive values, various distribution models, including normal, Weibull, gamma, exponential, Cauchy, and BirSau distributions, can be employed for modeling. To achieve this, we performed analyses that involved calculating the minimum Akaike information criterion (AIC) and Bayesian information criterion (BIC) defined as

$$AIC = -2 \ln L + 2k, \tag{31}$$

and

$$BIC = -2 \ln L + 2k \ln(n), \tag{32}$$

respectively, where L is the likelihood function, k is the number of parameters, and n is the number of observations. These analyses help us determine the best-fitting distribution for the data. We may conclude that the BirSau distribution effectively fits these datasets, as indicated in Tables 4 and 5. The summary statistics for the wind speed data from two industrial areas, situated in Samut Prakan and Rayong Province, are presented in Table 6. The basic statistics were $(\hat{\alpha}_1, \hat{\alpha}_2) = (0.4908, 0.3049)$; $\left(\hat{\beta}_1, \hat{\beta}_2\right) = (2.2613, 1.6999)$; $(n_1, n_2) = (92, 92)$ and $\hat{\omega} = 1.3004$. Table 7 presents the 95% CIs for the wind speed data ratio from Samut Prakan and Rayong provinces in Thailand. Based on the simulation results from the previous section, we will compare the case with large sample sizes. This confirmation indicates that the BCI was suitable for constructing the CI for the wind speed in this dataset, as we found that its width was shorter than that of the other method.

Table 3: The daily wind speed data from Samut Prakan and Rayong provinces, Thailand.

Provinces	Wind speed (knots)								
Samut Prakan	2.0	2.1	3.3	1.0	1.5	1.2	2.6	1.2	2.1
	1.3	1.3	3.5	2.1	1.8	1.2	1.7	1.8	1.5
	1.5	1.6	2.0	3.2	2.7	0.7	1.1	1.5	1.6
	1.5	2.1	1.3	3.4	2.7	1.6	1.4	1.7	1.4
	0.9	1.6	1.2	3.4	3.0	2.7	3.2	2.0	1.6
	3.2	2.0	1.1	3.5	1.9	1.2	4.6	2.1	1.7
	2.9	1.1	3.0	3.3	4.1	1.2	5.1	2.3	1.4
	2.3	1.1	2.2	5.2	5.4	1.1	4.1	1.8	2.8
	1.6	1.8	2.3	3.2	5.4	1.7	2.7	1.1	1.5
	1.6	2.2	1.1	3.1	3.0	2.1	2.0	1.1	0.8
	1.5	1.4							
Rayong	1.3	2.0	1.7	1.0	1.3	2.0	0.9	1.2	3.1
	1.8	1.8	1.8	1.2	1.3	1.9	1.6	1.1	1.5
	1.3	1.5	1.7	1.8	1.3	1.3	1.3	1.0	1.5
	0.8	1.5	1.9	2.4	1.4	1.2	1.1	1.1	1.1
	1.8	1.6	2.0	1.7	1.5	1.4	1.6	1.7	1.2
	1.7	1.7	1.7	3.2	1.6	1.7	1.6	2.0	1.4
	1.3	1.7	1.9	3.0	1.9	1.8	2.5	2.8	1.9
	1.1	1.6	2.3	2.9	2.9	1.0	1.6	1.4	1.6
	1.4	1.5	2.1	2.1	3.3	1.3	1.4	1.2	1.0
	1.1	1.4	1.8	1.6	1.5	2.4	1.7	1.5	1.0
	1.4	2.1							

Table 4: AIC values for fitting the positive wind speed data from Samut Prakan and Rayong provinces, Thailand.

Provinces	Distributions						
	BirSau	Cauchy	Logistic	Exponential	Gamma	Weibull	Normal
Samut Prakan	**243.2365**	286.1387	271.5496	327.6814	248.5576	260.7358	276.4713
Rayong	**126.9060**	151.8474	139.0542	279.3508	129.5592	148.5398	146.6014

Table 5: BIC values for fitting the positive wind speed data from Samut Prakan and Rayong provinces, Thailand.

Provinces	Distributions						
	BirSau	Cauchy	Logistic	Exponential	Gamma	Weibull	Normal
Samut Prakan	**250.8018**	291.1823	276.5932	330.2032	253.6012	265.7793	281.5148
Rayong	**134.4720**	156.8909	144.0978	281.8726	134.6027	153.5834	151.6450

Table 6: Summary statistics for the wind speed data.

Province	n	Min	Median	Mean	Max	Variance
Samut Prakan	92	0.7	1.8	2.1598	5.4	1.1442
Rayong	92	0.8	1.6	1.6609	3.3	0.2789

Table 7: The 95% CIs for the ratio of the mean of wind speed data for the Samut Prakan and Rayong datasets.

Methods	**Samut Prakan /Rayong**		
	Lower	Upper	width
GCI	1.1597	1.4642	0.3045
BCI	1.1629	1.4343	**0.2715**
PBCI	1.1770	1.4715	0.2945
BayCrI	1.1641	1.4682	0.3040
HPD	1.1584	1.4583	0.2999

5 Conclusions

We utilized the GCI, BCI, PBCI, BayCrI, and HPD methods to construct CIs for the ratio between the means of two BirSau distributions. The performance of these CIs was assessed in terms of their CPs and AWs. The results of a simulation study indicate that, for small and medium sample sizes, the HPD method generally provided satisfactory performance across most of the scenarios considered. In the case of a large sample, the BCI method stood out as the most effective approach. Furthermore, we applied similar methods to generate CIs for the ratio of means in wind speed datasets from the Thailand provinces of Rayong and Samut Prakan, representing two distinct industrial zones. The results when we applied these methods to a real situation were very similar to what we found in our simulations. When we had large sample sizes, the BCI method worked better in terms of the AWs. In summary, the HPD method is a suitable choice for constructing CIs for the ratios of means when dealing with small and medium-sized samples. However, when handling a large sample, the BCI method is identified as the appropriate approach for constructing a CI for the ratio of means, particularly when the two population groups follow BirSau distributions. In this study, the BirSau distribution applies to data with positive values. However, if the data under consideration contains both zero and positive values, future research should concentrate on investigating confidence intervals for the parameter of the delta-BirSau distribution to address this issue.

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Conflicts of Interest The authors declare no conflict of interest.

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